Appendix C
Resource for Secondary School Teachers: Circumcenter, Orthocenter, and Centroid

The purpose of this appendix is to give a demonstration—albeit on a small scale—of how the usual tedium and pitfalls of the axiomatic development of Euclidean geometry might be avoided. It deals with very standard materials: why the perpendicular bisectors (resp., *altitudes* and *medians*) of a triangle must meet at a point, the *circumcenter* (resp., *orthocenter* and *centroid*) of the triangle. The exposition starts at the lowest level—the axioms—and ends with a proof of the concurrence (i.e., meeting at a point) of the medians. It also includes a collection of exercises on proofs as an indication of how and when such exercises might be given. The major theorems to be proved (Theorems 1 and 11–14) are all interesting and are likely regarded as surprising to most students. These theorems would therefore do well to hold students' attention and convince students of the value of mathematical proofs.

The goal of this appendix is to prove the concurrence of the medians. If one turns this proof "inside out," so to speak, one will get the proof of the concurrence of the altitudes. The proof of the latter theorem is also included as is a demonstration of the concurrence of the perpendicular bisectors since that is also needed. The fact that the concurrence of the angle bisectors (the *incenter*) is left out is therefore entirely accidental. This appendix makes no pretense at completeness because its only purpose is to demonstrate a particular approach to geometric exposition, but if it did, then certainly the four centers would have been discussed together.

We specifically call attention to the following features:

1. The appearance of the exercises on proofs is intentionally gauged to approximate at what point of the axiomatic development those exercises should be given to students in a classroom situation. The first of such exercises asks only for a straightforward imitation of a proof that has just been presented (Lemma (2B)). The next one asks only for the reasons for some steps in the proof of Lemma 6, and by then students have already been exposed to several nontrivial proofs. The first exercise that asks for a genuine proof occurs after Lemma 7. In other words, students are given ample time to absorb the idea of a proof by studying several good examples before they are asked to construct one themselves.

   A conscientious effort was also made to ensure that the exercises all have some geometric content so that any success with them would require some geometric understanding instead of just facility with formal reasoning.

2. Certain facts are explicitly assumed without proof before some of the proofs (*local axiomatics*). Students should be informed that they too can make use of these unproven assertions.

3. None of the results presented is trivial to a beginner (except Lemma 4). It is hoped that, altogether, these results will convince the students of the benefit of learning about proofs; namely, to understand why some interesting things are true and be able, in turn, to present arguments to convince other people. In fact, most students probably do not believe any of the major theorems (Theorems 1 and 11–14) before being exposed to their proofs.
4. The concurrence of the altitudes and medians (Theorems 11 and 14) is usually not presented in standard textbooks except by use of coordinate geometry or the concept of similarity. Thus those theorems tend not to appear in a typical geometry curriculum. However, the easy access to them as demonstrated by this appendix should be a convincing argument that, with a little effort, it is possible to present students with interesting results very early in a geometry course.
5. The two-column proofs given in the following pages most likely do not conform to the rigid requirements imposed on the students in some classrooms (cf. Schoenfeld 1988, 145–66). However, for exposition, they are perfectly acceptable by any mathematical standards. It is hoped that their informal character would help restore the main focus of a proof, which is the correctness of the mathematical reasoning instead of a rigidly correct exposition.
6. The proof of Theorem 12 is given twice: once in the two-column format and the second time in the narrative (paragraph) format. In the classroom such "double-proofs" should probably be done for a week or two to lead students away from the two-column format. The proof of Theorem 13 is given only in the narrative format.

Axioms

We shall essentially assume the School Mathematics Study Group (SMSG) axioms, which are *paraphrased* below rather than quoted verbatim for easy reference; the relevant definitions are usually omitted (see Cederberg 1989, 210–11). Only those axioms pertaining to *plane* Euclidean geometry are given. Moreover, a school geometry course has no time for a *minimum* set of axioms. The last three axioms have therefore been added to speed up the logical development.

1. Two points A and B determine a unique line, to be denoted by AB.
2. (The Distance Axiom). To every pair of distinct points there corresponds a unique positive number, called their *distance.* This distance satisfies the requirement of the next axiom.
3. (The Ruler Axiom). Every line can be put in one-one correspondence with the real numbers so that if P and Q are two points on the line, then the absolute value of the difference of the corresponding real numbers is the distance between them.
4. (The Ruler Placement Axiom). Given two points P and Q on a line, the correspondence with real numbers in the preceding axiom can be chosen so that P corresponds to zero and Q corresponds to a positive number.

5. There are at least three noncollinear points.
6. (The Plane Separation Axiom). Given a line ℓ. Then the points not on ℓ form two convex sets, and any line segment $\overline{AB}$ joining a point A in one set and a point B in the other must intersect ℓ. The convex sets are called the *half-planes* determined by ℓ.
7. (The Angle Measurement Axiom). To every $\angle ABC$ there corresponds a real number between 0 and 180, to be denoted by $m\angle ABC$, called the *measure* of the angle.
8. (The Angle Construction Axiom). Given a line AB and a half-plane H determined by AB, then for every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ in H so that $m\angle PAB = r$.
9. (The Angle Addition Axiom). If D is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.
10. (The Angle Supplement Axiom). If two angles form a linear pair, then their measures add up to 180.
11. SAS Axiom for congruence of triangles.
12. (The Parallel Axiom). Through a given external point, there is at most one line parallel to a given line.
13. (The Area Axiom). To every polygonal region, there corresponds a unique positive number, called its *area,* with the following properties: (i) congruent triangles have the same area; (ii) area is additive on disjoint unions; and (iii) the area of a rectangle is the product of the lengths of its sides.
14. SSS Axiom for congruence of triangles.
15. ASA Axiom for congruence of triangles.
16. (The AA Axiom for Similarity). Two triangles with two pairs of angles equal are similar.

Perpendicularity

In the following exposition, we shall denote both the line segment from point A to point B and the distance from A to B simply by $\overline{AB}$. In other words, $\overline{AB}$ will denote also the length of the line segment $\overline{AB}$. Similarly, we shall shorten the notation for the measure of an angle $m\angle ABC$ to just $\angle ABC$. Thus $\overline{AB} < \overline{CD}$ means $\overline{CD}$ is longer than $\overline{AB}$, and $\angle ABC = 45$ means angle ABC has 45 degrees.

Recall that $\angle DCB$ is a *right angle* (see figure 1), and DC is *perpendicular* to CB, if for a point A collinear with C and B and on the other side of C from B, $\angle DCA = \angle DCB$. If $\angle DCB$ is a right angle, then its measure is 90 because by the Angle Supplement Axiom $\angle DCA + \angle DCB = 180$ so that $\angle DCB + \angle DCB = 180$, and we obtain $\angle DCB = 90$.

Similarly, $\angle DCA = 90$. Conversely, if A, C, B are collinear and $\angle DCB = 90$, the same argument shows $\angle DCA = 180 - \angle DCB = 90$, and $\angle DCB = \angle DCA$ so that $\angle DCB$ is a right angle. Thus we can assert that two lines ℓ_1, ℓ_2 are perpendicular if one of the angles they form is 90.

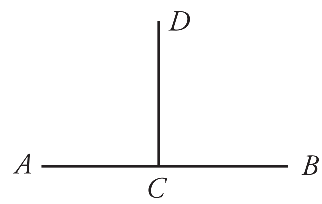

Figure 1

Recall also that a point C on a segment $\overline{AB}$ is the *midpoint* of $\overline{AB}$ if $\overline{AC} = \overline{CB}$. (Recall the *convention:* AB denotes the line containing A and B, and $\overline{AB}$ denotes the line *segment* joining A to B.) The straight line passing through the midpoint of a segment and perpendicular to it is called the *perpendicular bisector* of the segment. Note that every segment has a perpendicular bisector. Indeed, given $\overline{AB}$, the Ruler Axiom guarantees that there is a midpoint C of $\overline{AB}$, and the Angle Construction Axiom guarantees that there is an $\angle DCB = 90$. Then by the preceding discussion, $DC \perp AB$, and DC is the perpendicular bisector.

We shall need the fact that the perpendicular bisector of a segment is unique; that is, if DC and $D'C'$ are perpendicular bisectors of $\overline{AB}$, then $DC = D'C'$. This is so intuitively obvious that we shall not spend time to prove it.

[For those interested in a proof, however, one uses the Distance Axiom and the Ruler Axiom to show that the midpoint of a segment must be unique, and then one uses the Angle Construction Axiom to show that the line passing through the midpoint and perpendicular to the segment is also unique.]

Three lines are *concurrent* if they meet at a point. The following gives a surprising property about perpendicular bisectors:

Theorem 1. *The perpendicular bisectors of the three sides of a triangle are concurrent.* (The point of concurrency is called the *circumcenter* of the triangle.)

Let $A'B'C'$ be the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively (see figure 2). A naive approach would try to prove directly that all three perpendicular bisectors meet at a point O. This is too clumsy and also unnecessary. A better way is the following: Take two of the perpendicular bisectors, say, those at A' and B', and let them meet at a point O. Then we show that O must lie on the perpendicular bisector of $\overline{AB}$. Theorem 1 would be proved.

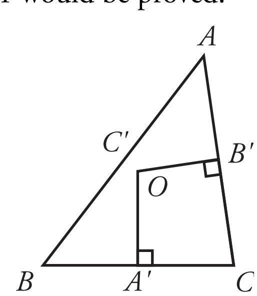

Figure 2

One would appreciate this approach to the proof of Theorem 1 more if the perpendicular bisector of a segment is better understood. To this end we first prove:

Lemma 2. *A point D is on the perpendicular bisector of a segment* $\overline{AB}$ *if and only if* $\overline{DA} = \overline{DB}$.

Proof. First, we explain the "if and only if" terminology. It is a shorthand to indicate that two assertions must be proved:

(i) *If the statement preceding this phrase is true, then the statement following this phrase is also true.*

(ii) *If the statement following this phrase is true, then the statement preceding this phrase is also true.*

For the case at hand, this means we have to prove two things (see figure 3):

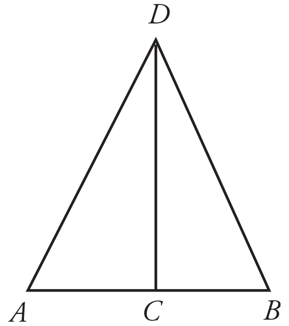

Figure 3

Lemma (2A). *If D is on the perpendicular bisector of* $\overline{AB}$*, then* $\overline{DA} = \overline{DB}$.

Lemma (2B). *If* $\overline{DA} = \overline{DB}$*, then D is on the perpendicular bisector of* $\overline{AB}$.

Proof of (2A): Let CD be the perpendicular bisector of $\overline{AB}$.

1. $\overline{AC} = \overline{CB}$, and $\angle DCA = \angle DCB = 90$.	1. Hypothesis.
2. $\overline{CD} = \overline{CD}$.	2. Obvious.
3. $\triangle ADC \cong \triangle BDC$.	3. SAS.
4. $\overline{DA} = \overline{DB}$.	4. Corresponding sides of congruent triangles. Q.E.D.

Proof of (2B): Given that $\overline{DA} = \overline{DB}$, we have to show that the perpendicular bisector of $\overline{AB}$ passes through D. Instead of doing so directly, we do something rather clever: we are going to construct the *angle bisector CD* of $\angle ADB$. This means $\angle ADC = \angle CDB$. Of course, we must first prove that there is such a line CD with the requisite property. Then we shall prove $CD \perp AB$ and $\overline{AC} = \overline{CB}$ so that CD *is* the perpendicular bisector of $\overline{AB}$.

Recall that the Plane Separation Axiom makes it possible to define the *interior* of $\angle ADB$ as the intersection of the half-plane determined by DA which contains

B and the half-plane determined by *DB* which contains *A*. Now the following assertion is obvious pictorially:

Assertion A. *If E is a point such that* $\angle ADE < \angle ADB$, *then E is in the interior of* $\angle ADB$. *Furthermore, the line DE intersects* $\overline{AB}$.

For those who are truly curious, let it be mentioned that the first part of Assertion A can be proved by using the Angle Addition Axiom and the second part by repeated applications of the Plane Separation Axiom (see figure 4).

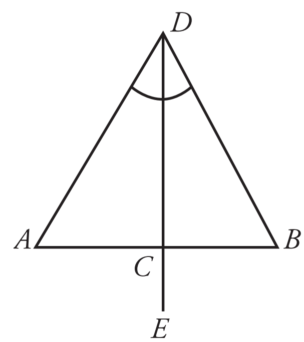

Figure 4

1. If the measure of $\angle ADB$ is x, there is a point E in the interior of $\angle ADB$ so that $\angle ADE = \frac{x}{2}$.	1. Angle Construction Axiom and Assertion A.
2. If DE meets $\overline{AB}$ at C, $\angle ADC + \angle CDB = \angle ADB = x$.	2. Angle Addition Axiom and Assertion A.
3. $\angle CDB = x - \angle ADC = x - \frac{x}{2} = \frac{x}{2}$.	3. By 1 and 2.
4. $\angle ADC = \angle CDB$.	4. By 1 and 3.
5. $\overline{DA} = \overline{DB}$.	5. Hypothesis.
6. $\overline{CD} = \overline{CD}$.	6. Obvious.
7. $\triangle ACD \cong \triangle BCD$.	7. SAS.
8. $\overline{AC} = \overline{CB}$ and $\angle DCA = \angle DCB$.	8. Corresponding angles and sides of congruent triangles.
9. CD is the perpendicular bisector of $\overline{AB}$.	9. By 8 and the definition of a perpendicular bisector. Q.E.D.

Exercise 1. Using the preceding proof as a model, write out a complete proof of the fact that if $\triangle ABC$ is isosceles with $\overline{AB} = \overline{AC}$, then the angle bisector of $\angle A$ is also the perpendicular bisector of $\overline{BC}$.

From the point of view of Lemma 2, our approach to the proof of Theorem 1 is now more transparent. As mentioned previously, this approach is to let the perpendicular bisectors $A'O$ and $B'O$ of $\overline{BC}$ and $\overline{AC}$, respectively, meet at a point O. Then we shall prove that O lies on the perpendicular bisector of $\overline{AB}$ (see figure 5).

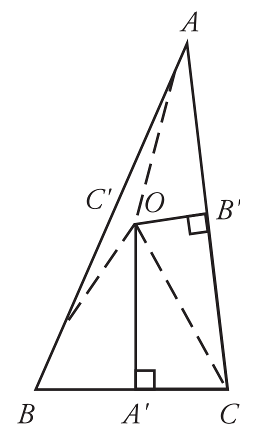

Figure 5

Proof of Theorem 1. Join OB, OC, and OA.

1. $\overline{OB} = \overline{OC}$.	1. Lemma (2A) and the fact that $\overline{OA'}$ is the perpendicular bisector of $\overline{BC}$.
2. $\overline{OC} = \overline{OA}$.	2. Lemma (2A) and the fact that $\overline{OB'}$ is the perpendicular bisector of $\overline{AC}$.
3. $\overline{OB} = \overline{OA}$.	3. From 1 and 2.
4. O lies on the perpendicular bisector of $\overline{AB}$.	4. Lemma (2B). Q.E.D.

Circumcenter of a Triangle

Corollary to Theorem 1. *The circumcenter of a triangle is equidistant from all three vertices.*

The corollary is obvious if we look at steps 1–3 of the preceding proof. Note that, as figure 6 suggests, the circumcenter can be in the exterior of the triangle.

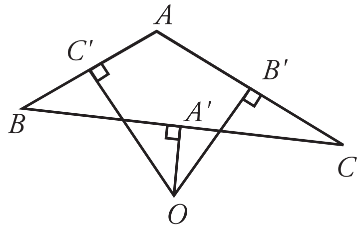

Figure 6

Several remarks about the proof of Theorem 1 are in order. First of all, this proof suggests a general method of proving the concurrency of three lines: let two of the three meet at a point O, and then show that O must lie on the third line. Technically, this is easier than directly proving that all three lines meet at a point. This method is a kind of *indirect proof* and is useful in many situations; for example, in proving that the three angle bisectors of a triangle are concurrent. In other words, if we think of proving a theorem as fighting a battle against an enemy, then it makes sense that *sometimes* we can defeat the enemy without resorting to a frontal attack.

A second remark has to do with the tacit assumption discussed previously; namely, that the perpendicular bisectors OA' and OB' of $\overline{BC}$ and $\overline{AC}$, respectively, do meet at a point O. This is obvious from figure 6, and we usually do not bother to prove such obvious statements, being fully confident that—if challenged—we can prove them. For the sake of demonstration, however, we will supply a proof this time after we have proved a few properties of parallel lines. Thus, we shall prove:

Assertion B. *Perpendicular bisectors from two sides of a triangle must intersect.*

Note that there is no circular reasoning here: Assertion B will not be used to prove any of the theorems involving parallel lines. Indeed, we shall not have to face Assertion B again in this appendix.

A third remark concerns the name *circumcenter*. A *circle* with center O and radius r is by definition the collection of all points whose distance from O is r. The corollary to Theorem 1 may then be rephrased as: the circle with center O and radius $\overline{OA}$ passes through all three vertices. This circle is called the *circumcircle* of $\triangle ABC$, which then gives rise to the name "circumcenter." (*Circum* means "around.") Incidentally, Theorem 1 proves that any triangle determines a circle that passes through all three vertices.

Next, we turn attention to the *altitudes* of a triangle; that is, the perpendiculars from the vertices to the opposite sides (see figure 7). We want to show that they too are concurrent. This demonstration needs some preparation. First of all, we have to show that altitudes exist; that is, through each vertex there is a line that is perpendicular to the opposite side. More generally, we shall prove:

Lemma 3. *Given a point P and a line ℓ not containing P, there is a line PQ which is perpendicular to ℓ.*

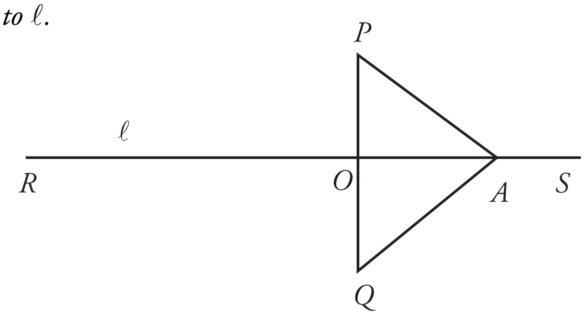

Figure 7

Proof. Let $\ell = RS$. Recall that ℓ contains an infinite number of points (the Ruler Axiom) and that the Plane Separation Axiom allows us to talk about the two sides of ℓ.

1. Join P to an arbitrary point A on ℓ.	1. Two points determine a line.
2. If x is the measure of $\angle PAR$, let Q be a point on the side of ℓ not containing P so that $\angle PAQ = x$.	2. The Angle Construction Axiom.
3. We may let Q be the point on AQ so that Q, P are on opposite sides of ℓ and $\overline{AQ} = \overline{AP}$.	3. The Distance Axiom.
4. PQ meets ℓ at some point O.	4. The Plane Separation Axiom.
5. $\angle PAO = \angle QAO$.	5. From 1 and 2.
6. $\overline{OA} = \overline{OA}$.	6. Obvious.
7. $\triangle PAO \cong \triangle QAO$.	7. SAS.
8. $\angle AOP = \angle AOQ$.	8. Corresponding angles of congruent triangles.
9. PQ $\perp$ ℓ.	9. By definition of perpendicularity. Q.E.D.

Vertical Angles

Before we turn to parallel lines, we do some spadework. The teacher introduces the definition of *vertical angles* (omitted here).

Lemma 4. *Vertical angles are equal.*

Proof. Let AB, CD meet at O. We will show $\angle AOD = \angle BOC$ (see figure 8).

1. $\angle AOD + \angle DOB = 180$ and $\angle DOB + \angle BOC = 180$.	1. The Angle Supplement Axiom.
2. $\angle AOD + \angle DOB = \angle DOB + \angle BOC$.	2. By 1.
3. $\angle AOD = \angle BOC$.	3. From 2 and the cancellation law of addition. Q.E.D.

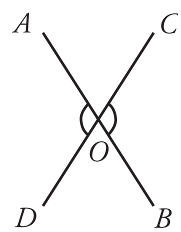

Figure 8

The teacher introduces the definitions of *exterior angle* and *remote interior angles* of a triangle (omitted here). To prove the next proposition, we shall assume a geometrically obvious fact. In figure 9 if M is any point on $\overline{AC}$, we shall assume as known:

Assertion C. *If we extend* $\overline{BM}$ *along M to a point E, then E is always in the interior of* $\angle ACD$.

This can be proved with repeated applications of the Plane Separation Axiom, but the argument is not inspiring.

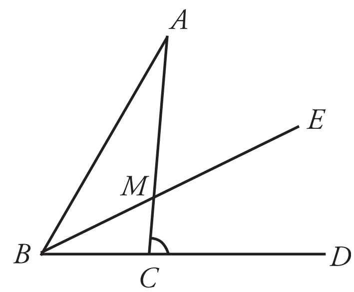

Figure 9

Proposition 5. *An exterior angle of a triangle is greater than either remote interior angle.*

Proof. Let us show $\angle ACD > \angle BAC$ (see figure 10). To show $\angle ACD > \angle ABC$, we observe that the same proof would show $\angle BCG > \angle ABC$ and then use Lemma 4 to get $\angle BCG = \angle ACD$. Putting the two facts together, we get $\angle ACD > \angle ABC$.

Join B to the midpoint M of $\overline{AC}$ and extend BM to a point E such that $\overline{BM} = \overline{ME}$ (the Ruler Axiom). Join CE.

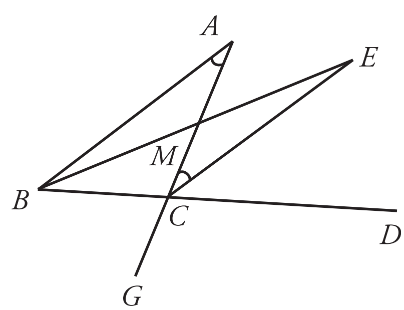

Figure 10

1. $\angle AMB = \angle EMC$.	1. Lemma 4.
2. $\overline{BM} = \overline{ME}$.	2. By construction.
3. $\overline{AM} = \overline{MC}$.	3. M is the midpoint of $\overline{AC}$.
4. $\triangle AMB \cong \triangle CME$.	4. SAS.
5. $\angle BAM = \angle MCE$.	5. Corresponding angles of congruent triangles.
6. $\angle MCE < \angle ACD$.	6. By the Angle Addition Axiom and Assertion C.
7. $\angle BAC < \angle ACD$.	7. By 5 and 6. Q.E.D.

Parallel Lines

We now come to some basic facts about parallel lines. Given two lines ℓ_1 and ℓ_2, one can introduce the definition of *alternate interior angles* and *corresponding angles* of ℓ_1 and ℓ_2 with respect to a transversal (omitted here). We shall need:

Lemma 6. *If two lines make equal alternate interior angles with a transversal, they are parallel.*

Proof. Let the transversal be BE. Designate the two equal alternate interior angles as $\angle\alpha$ and $\angle\beta$ (see figure 11). We assume that AC is not $||$ to DF and deduce a contradiction. (This is an example of *proof by contradiction*.)

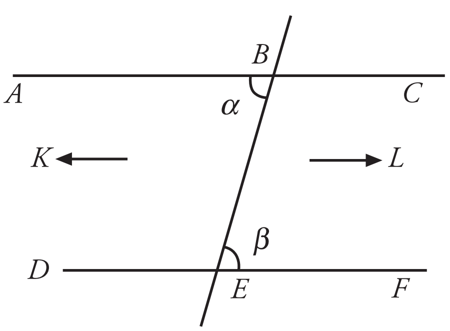

Figure 11

1. AC meets DF either at a point K to the left of BE or at L to the right of BE.	1. By the fact that AC is not $||$ to DF and by the Plane Separation Axiom.
2. If AC meets DF at K, then $\angle\beta > \angle\alpha$, contradicting $\angle\alpha = \angle\beta$.	2. $\boxed{}$
3. If AC meets DF at L, then $\angle\alpha > \angle\beta$, also contradicting $\angle\alpha = \angle\beta$.	3. $\boxed{}$ Q.E.D.

Exercise 2. Supply the reasons for steps 2 and 3.
[Answers: Proposition 5 is the reason for both.]

Note: The textbook and teacher must make sure that students are eventually given the answers to problems of this nature; it is important to bring closure to a mathematical discussion.

This proposition complements the parallel axiom in the following sense. Notation is given as in the preceding proof: suppose that *DF* and *B* are given and we want to construct a line through *B* and $\|$ *DF*. By the Angle Construction Axiom, with $\angle\beta$ as given, we can construct $\angle\alpha$ with vertex at the given *B* on the other side of $\angle\beta$ but with the same measure. Then by Lemma 6, *AC* is a line passing through *B* which is $\|$ *DF*. Therefore:

Corollary to Lemma 6. *Through a point not on a line ℓ, there is one and only one line parallel to ℓ.*

Lemma 7 is the converse of Lemma 6.

Lemma 7. *Alternate interior angles of parallel lines with respect to a transversal are equal.*

Proof. The notation is as before, suppose $AC \parallel DF$. We shall prove $\angle\alpha = \angle\beta$ (see figure 12).

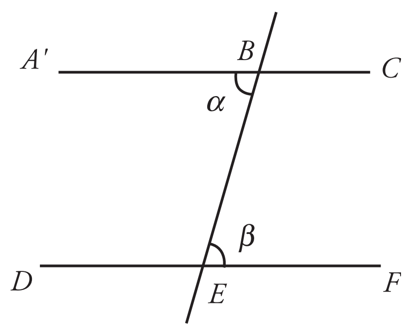

Figure 12

1. At *B*, construct $\angle A'BE$ to the left of *BE* so that $\angle A'BE = \angle\beta$.	1. Angle Construction Axiom.
2. $A'B \parallel DF$.	2. $\boxed{}$
3. Since $A'B$ passes through *B*, $A'B = AB$.	3. $\boxed{}$
4. $\angle\alpha = \angle A'BE = \angle\beta$.	4. By 3. Q.E.D.

Exercise 3. Supply the reasons for steps 2 and 3.
[Answers: Step 2. Lemma 6. Step 3. The Parallel Axiom.]

Exercise 4. Prove that corresponding angles of parallel lines with respect to a transversal are equal.

Parallelograms

The teacher introduces the definition of a *quadrilateral* (omitted here). A *parallelogram* is a quadrilateral with parallel opposite sides. We shall need two properties of parallelograms that are pictorially plausible when a parallelogram is drawn carefully.

Lemma 8. *A quadrilateral is a parallelogram if and only if it has a pair of sides which are parallel and equal.*

Lemma 9. *A quadrilateral is a parallelogram if and only if its opposite sides are equal.*

Proof of Lemma 8. First we prove that if quadrilateral $ABCD$ has a pair of sides which are parallel and equal, then it is a parallelogram. In figure 13 we assume $\overline{AB} = \overline{CD}$ and $AB \parallel CD$. Then we have to prove $AD \parallel BC$.

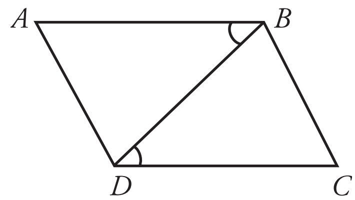

Figure 13

1. Join BD. $\overline{BD} = \overline{BD}$.	1. Two points determine a line.
2. $\angle ABD = \angle BDC$.	2. Lemma 7.
3. $\triangle ABD \cong \triangle CDB$.	3. SAS.
4. $\angle ADB = \angle DBC$.	4. Corresponding angles of congruent triangles.
5. $AD \parallel BC$.	5. Lemma 6. Q.E.D.

Next we prove that a parallelogram has a pair of sides which are parallel and equal. Since $AB \parallel DC$ by definition, it suffices to prove that $\overline{AB} = \overline{DC}$. Let notation be as in the preceding proof.

1. $\angle ABD = \angle BDC$ and $\angle ADB = \angle DBC$.	1. Lemma 7.
2. $\overline{BD} = \overline{BD}$.	2. Obvious.
3. $\triangle ABD \cong \triangle CDB$.	3. ASA.
4. $\overline{AB} = \overline{DC}$.	4. Corresponding sides of congruent triangles. Q.E.D.

Exercise 5. Prove Lemma 9 (it is similar to the proof of Lemma 8).

Exercise 6. Prove that a quadrilateral *ABCD* is a parallelogram if and only if the *diagonals* $\overline{AC}$ and $\overline{BD}$ bisect each other; that is, if they intersect at *E*, then $\overline{AE} = \overline{EC}$ and $\overline{BE} = \overline{ED}$ (see figure 14).

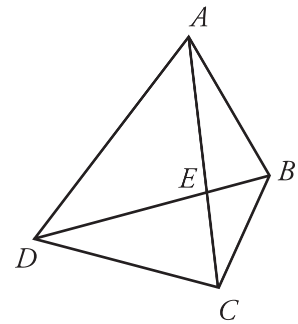

Figure 14

The following lemma is an immediate consequence of Lemma 6. Convince yourself of this and be sure to draw pictures to see what it says.

LEMMA 10. *Suppose two lines* ℓ_1 *and* ℓ_2 *are parallel.* (i) *If* ℓ *is a line perpendicular to* ℓ_1, *then* ℓ *is also perpendicular to* ℓ_2. (ii) *If another two lines* L_1 *and* L_2 *satisfy* $L_1 \perp \ell_1$ *and* $L_2 \perp \ell_2$, *then* $L_1 \parallel L_2$.

We can now prove Assertion B stated after the proof of Theorem 1. Let $\triangle ABC$ be given and let lines ℓ_1 and ℓ_2 be the perpendicular bisectors of $\overline{BC}$ and $\overline{AC}$, respectively, (see figure 15). Let lines L_1 and L_2 be lines containing $\overline{BC}$ and $\overline{AC}$, respectively. If Assertion B is false, then $\ell_1 \parallel \ell_2$. By Lemma 10 (ii), $L_1 \parallel L_2$. But we know L_1 meets L_2 at *C*, a contradiction. Thus ℓ_1 must meet ℓ_2 after all.

We are in a position to prove one of our main results.

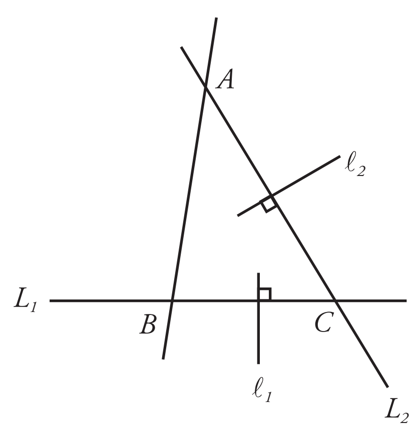

Figure 15

The Orthocenter

Theorem 11. *The three altitudes of a triangle are concurrent.* (This point is called the orthocenter of the triangle.)

Proof. Let $\triangle ABC$ be given and let its altitudes be AD, BE, and CF. The idea of the proof is to turn AD, BE, CF into perpendicular bisectors of a bigger triangle and use Theorem 1. The idea itself is sophisticated and is attributed to the great mathematician C. F. Gauss. Technically, however, it is quite simple to execute. It illustrates a general phenomenon in mathematics: sometimes a seemingly difficult problem becomes simple when it is put into the proper context (see figure 16).

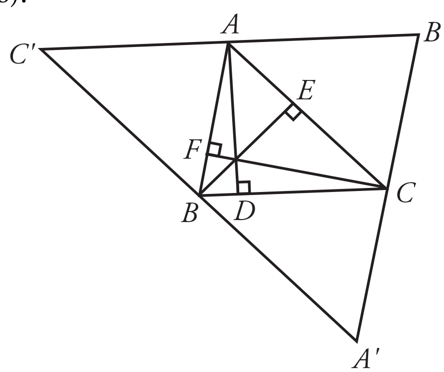

Figure 16

1. Through A, there is a line $C'B' \parallel BC$.	1. Corollary to Lemma 6.
2. Similarly, let $C'A'$ and $B'A'$ be lines through B and C, respectively, such that $C'A' \parallel AC$ and $B'A' \parallel AB$.	2. Same reason.
3. $ABCB'$ is a parallelogram.	3. From 1 and 2.
4. $\overline{AB'} = \overline{BC}$.	4. Lemma 9.
5. $ACBC'$ is likewise a parallelogram and $C'A = \overline{BC}$.	5. See 3 and 4.
6. $\overline{C'A} = \overline{AB'}$.	6. From 4 and 5.
7. $AD \perp BC$.	7. Hypothesis.
8. $AD \perp C'B'$.	8. Lemma 10(i).
9. AD is the perpendicular bisector of $\overline{C'B'}$.	9. From 6 and 8.
10. Similarly, BE and CF are perpendicular bisectors of $\overline{C'A'}$ and $\overline{A'B'}$, respectively.	10. See 3 through 9.
11. AD, BE, and CF are concurrent.	11. Apply Theorem 1 to $\triangle A'B'C'$. Q.E.D.

The Medians and Centroid of a Triangle

The line joining a vertex of a triangle to the midpoint of the opposite side is called a *median* of the triangle.

Exercise 7. In the notation of the proof of Theorem 11, prove that AA', BB', and CC' are the medians of $\triangle ABC$ as well.

Finally, we turn to the proof of the concurrence of the medians. The proof will be seen to have many points of contact with the proof of Theorem 11 shown previously. Instead of turning "outward" to a bigger triangle, however, the proof of the concurrence of the medians turns "inward" and looks at the triangle obtained by joining the midpoints of the three sides. To this end, the following theorem is fundamental:

Theorem 12. *The line segment joining the midpoints of two sides of a triangle is parallel to the third side and is equal to half of the third side.*

Proof. Thus if $\overline{AE} = \overline{EB}$ and $\overline{AF} = \overline{FC}$, then $EF \parallel BC$ and $\overline{EF} = \frac{1}{2}\overline{BC}$.

To motivate the proof, note that all the axioms and the theorems presented so far deal with the equality of two objects (angles, segments, and so forth), not about *half* of something else. So it makes sense to try to reformulate $\overline{EF} = \frac{1}{2}\overline{BC}$ as a statement about the equality of two equal segments (see figure 17). What then is simpler than doubling $\overline{EF}$? Students will learn that the construction of so-called auxiliary lines, such as $\overline{FP}$ and $\overline{PC}$ in the following proof, is a fact of life in Euclidean geometry.

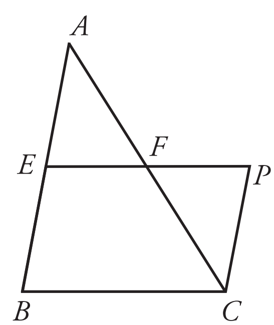

Figure 17

1. Extend $\overline{EF}$ along F to P so that $\overline{EF} = \overline{FP}$ and join PC.	1. The Ruler Axiom and the fact that two points determine a line.
2. $\overline{AF} = \overline{FC}$.	2. Hypothesis.
3. $\angle AFE = \angle PFC$.	3. Lemma 4.
4. $\triangle AFE \cong \triangle CFP$.	4. $\boxed{}$
5. $\angle AEF = \angle FPC$.	5. By 4 and corresponding angles of congruent triangles are equal.
6. $EB \parallel PC$.	6. Lemma 6.

7. $\overline{AE} = \overline{PC}$.	7. By 4 and corresponding sides of congruent triangles are equal.
8. $\overline{AE} = \overline{EB}$.	8. Hypothesis.
9. $\overline{EB} = \overline{PC}$.	9. By 7 and 8.
10. *EBCP* is a parallelogram.	10. $\boxed{}$
11. $EF \parallel BC$.	11. By 10 and the definition of a parallelogram.
12. $\overline{EP} = \overline{BC}$.	12. $\boxed{}$
13. $\overline{EP} = 2\overline{EF}$.	13. By 1.
14. $\overline{EF} = \frac{1}{2}\overline{BC}$.	14. By 12 and 13. Q.E.D.

Exercise 8. Supply the reasons for steps 4, 10, and 12.

[Answers: Step 4. By 1 through 3 and SAS. Step 10. By 6, 9, and Lemma 8. Step 12. By 10 and Lemma 9.]

Proof of Theorem 12 in narrative form. Extend $\overline{EF}$ to a point P so that $\overline{EF} = \overline{FP}$. Join PC. We are going to prove that $\triangle AEF \cong \triangle CPF$. This proof is possible because the vertical angles $\angle AFE$ and $\angle CFP$ are equal, $\overline{AF} = \overline{FC}$ by hypothesis and $\overline{EF} = \overline{FP}$ by construction. So SAS gives the desired congruence. It follows that $\angle AEF = \angle CPF$ and therefore that $AB \parallel PC$ (Lemma 6) and $\overline{AE} = \overline{PC}$. Because $\overline{AE} = \overline{EB}$, $\overline{EB}$ and $\overline{PC}$ are both parallel and equal. Hence *EBCP* is a parallelogram (Lemma 8). In particular, $EP \parallel BC$ and, by Lemma 10, $\overline{EP} = \overline{BC}$. Hence $\overline{BC} = \overline{EF} + \overline{FP} = 2\overline{EF}$. Q.E.D.

Exercise 9. Prove that two lines that are each parallel to a third line are parallel to each other.

In the next four exercises, do not use Axiom 16 on similarity (p. 289) *in your proofs.*

Exercise 10. Let E be the midpoint of $\overline{AB}$ in $\triangle ABC$. Then the line passing through E which is parallel to BC bisects AC.

Exercise 11. Let *ABCD* be *any* quadrilateral, and let A', B', C', and D' be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Then $A'B'C'D'$ is a parallelogram.

Exercise 12. Given $\triangle ABC$. Let L, M be points on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{AL} = \frac{1}{4}\overline{AB}$ and $\frac{1}{4}\overline{AB}$ and $\overline{AM} = \frac{1}{4}\overline{AC}$. Prove that $LM \parallel BC$ and $\overline{LM} = \frac{1}{4}\overline{BC}$.

Exercise 13. Given $\triangle ABC$. Let L, M be points on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{AL} = \frac{1}{3}\overline{AB}$ and $\overline{AM} = \frac{1}{3}\overline{AC}$. Prove that $LM \parallel BC$ and $\overline{LM} = \frac{1}{3}\overline{BC}$. (Hint: Begin by imitating the proof of Theorem 12.)

Exercise 14. (For those who know mathematical induction.) Let n be a positive integer. Given $\triangle ABC$. Let L, M be points on $\overline{AB}$ and $\overline{AC}$, respectively, so that $\overline{AL} = \frac{1}{n}\overline{AB}$ and $\overline{AM} = \frac{1}{n}\overline{AC}$. Prove that $LM \parallel BC$ and $\overline{LM} = \frac{1}{n}\overline{BC}$.

For the proof of the next theorem, we shall assume the following three facts:

Assertion D. *Any two medians of a triangle meet in the interior of the triangle.*

Assertion E. *Two lines that are each parallel to a third are parallel to each other.*

Assertion F. *The diagonals of a parallelogram bisect each other.*

For Assertion D one first has to define the *interior* of a triangle by using the Plane Separation Axiom. The proof then uses this axiom repeatedly, a tedious process. Assertion E is Exercise 9 shown previously, and Assertion F is Exercise 6 on page 300.

Theorem 13. *Let BE be a median of $\triangle ABC$. Then any other median must meet $\overline{BE}$ at the point G so that $\overline{BG} = 2\overline{GE}$.*

Proof. Let CF be another median and let CF meet BE at a point to be denoted also by G for simplicity. We will prove that $\overline{BG} = 2\overline{GE}$, which would then finish the proof of the theorem (see figure 18).

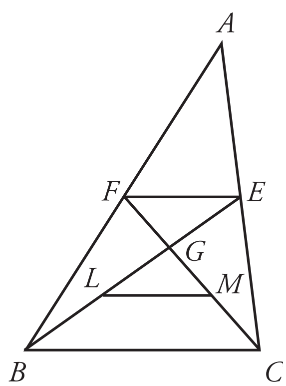

Figure 18

Join FE and join the midpoint L of $\overline{BG}$ to the midpoint M of $\overline{CG}$. Applying Theorem 12 to $\triangle ABC$, we get $\overline{BC} = 2\overline{FE}$ and $FE \parallel BC$. Similarly, applying the same theorem to $\triangle GBC$ yields $\overline{BC} = 2\overline{LM}$ and $LM \parallel BC$. Hence $\overline{FE}$ and $\overline{LM}$ are equal and parallel (by Assertion E), and $FEML$ is a parallelogram (Lemma 8). By Assertion F, $\overline{LG} = \overline{GE}$. L being the midpoint of $\overline{BG}$ implies $\overline{BL} = \overline{LG} = \overline{GE}$ so that $\overline{BG} = 2\overline{GE}$. Q.E.D.

Exercise 15. Let D, E, F be the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, in $\triangle ABC$. Prove that $\triangle AFE$, $\triangle DFE$, $\triangle FBD$, and $\triangle EDC$ are all congruent.

$\triangle DFE$ in Exercise 15 is called the *medial triangle* of $\triangle ABC$.

It follows immediately from Theorem 13 that both of the other two medians of $\triangle ABC$ must intersect $\overline{BE}$ at the point G. Hence we have:

Theorem 14. *The three medians of a triangle are concurrent, and the point of concurrency is two-thirds of the length of each median from the vertex.* (This is the *centroid* of the triangle.)

Exercise 16. Show that the centroid of a triangle is also the centroid of its medial triangle.

One may conjecture in view of Theorem 14 that if we trisect each side of a triangle, then the lines joining a vertex to an appropriate point of trisection on the opposite side may also be concurrent. One accurately drawn picture (see figure 19) is enough to lay such wishful thinking to rest. Such a picture then provides a *counterexample* to this conjecture.

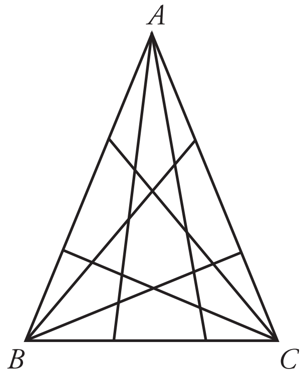

Figure 19

Part of the charm of Euclidean geometry is that most conjectures can be made plausible or refuted by a judicious picture. Compared with other subjects, such as algebra or calculus, this way of confronting a conjecture in geometry is by far the most pleasant.